Series

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Outline

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- Power Series
- Taylor Series

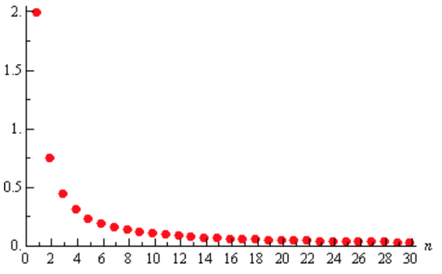
Sequences

- A sequence is an infinite list of numbers written in a definite order, e.g. $2, 4, 8, 16, 32, \ldots$
- Ther numbers in the list are called the **terms** of the sequence.
- A sequence is obtained by assigning to each positive integer n a number z_n – this is called the term of sequence. This can be written as $z_1, z_2, ...$ or just $\{z_n\}$
- A convergent sequence $z_1, z_2, ...$ is one that has a limit $c$as $\lim_{n \to \alpha} z_n = c$
- By definition, for every $\varepsilon > 0$ we can find an N such that $|z_n - c| < \varepsilon$
- A divergent sequence does not converge

Sequence

Find the first few terms of $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$

- To find the first few sequence, we just substitute the values of
- $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty} = \left\{2, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \ldots\right\}$
- For plot for the first 30 terms $(1,2), \left(2, \frac{3}{4}\right), \left(3, \frac{4}{9}\right), \ldots$



- As n approaches infinity, the sequence converge to a limit $\lim_{n\to\infty} \frac{n+1}{n^2} = 0$
- This sequence is convergent.
- The sequence $\{i^n = \{i, -1, -i, 1, \ldots\}\}$ is divergent

Sequence

Example

Arithmetic sequences

If $\{a_n\}$ is an arithmetic sequence with commond difference d, then $a_n = k + nd$ for some constant k, e.g. $\{5, 8, 11, 14, 17, \ldots\}$. The common difference is 3, the formula for this arithmetic sequence is $\{2 + 3n\}$

Example

Geometric sequence

If $\{a_n\}$ is a geometric sequence with a commond ratio r, then $a_n = kr^n$ for some constant k, e.g. $\{6, 12, 24, 48, 96, \ldots\}$.Since the common ratio is 2, the formula for this geometric sequence is $\{3 \cdot 2^k\}$

Series

- A series is an infinite sum of numbers: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots+\frac{1}{2^n}+\ldots$ The individual numbers are called the **terms** of the series. Using the summation notation, the series can be written as $\sum_{n=1}^{\infty} \frac{1}{2^n}$
- Given a sequence $z_1, z_2, z_3, \ldots, z_m, \ldots$, we may form the sequence of the sums $s_1 = z_1$, $s_1 = z_1 + z_2$, $s_1 = z_1 + z_2 + z_3, \ldots$
- In general, $s_n = z_1 + z_2 + \ldots + z_n$, $(n = 1, 2, \ldots)$
- s_n is called the **nth partial sum** of the *infinite series* or *series* $\sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots$
- A convergent series is one whose sequence of partial sums converges, $\lim_{n\to\infty} s_n = s$, then we can write $s = \sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots$
- If the sum is infinite or does not exist, then the series diverges
- In summary, if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges. If if $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ may converge or may diverge

Series

Example

Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$

The first few partial sum of the series are as follows

$s_1 = \frac{1}{2}$

$s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

The partial sums are converging to 1

The sum of an infinite series,

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ converges or diverges

Solution

$\lim_{n \to \infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$

This series diverges to ∞

Series.

- A geometric series is the sum of the terms of a geometric sequence.
- For example, the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$ The formula for a geometric series is $\sum_{n=0}^{\infty} ar^n$, where a is the first term and r is the common ratio
- The geometric series converges for $|r| < 1$and diverges for $|r| \geq 1$

Example

Rewrite the series $3 + 2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \cdots$ using the summation notation
Solution The first term of the series is 3, and each term is $\frac{2}{3}$ of the previous term, therefore it can be rewrite as $\sum_{n=0}^{\infty} 3 \cdot \left(\frac{2}{3}\right)^n$

Series.

- The sum of a geometric series, if $|r| < 1$ then $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

Example

(a) $\sum_{n=0}^{\infty} \frac{5}{3^n}$ (b) $8 + 6 + \frac{9}{2} + \frac{27}{8} + \frac{81}{32} + \cdots$

Convergence test

- Find the sum of a series is hard, so we focus on an easier problem: can we figure out whether a given series converges?
- Let foucs on a positive series, whose terms are all positive numbers
- For positive series, if $\sum a_n$ is a positive series, then either $\sum a_n$ converges to a positive number or $\sum a_n$ diverges to infinity
- If you want to know whether a series $\sum a_n$ with positive terms converges, then we need to know how quickly the term $a_n$approach zero as $n \to \infty$

Convergence test

- The comparison theorem: Let $\sum a_n$ and $\sum b_n$ be positive series, and suppose that $a_n \leq b_n$ for each term, then $\sum a_n \leq \sum b_n$

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges or diverges

Power Series

- A power series is a polynomial with infinitely many terms, for example, $f(x) = 1 + x + x^2 + x^3 + \cdots$
- Similar to a polynomial, a power series is a **function of x,** e.g. $f(\frac{1}{2}) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$
- The entire power series $1 + x + x^2 + x^3 + \cdots$ can be thought of as a geometric series with a common ratio of x, which gives a simple for the sum as $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$
- A power series that adds up to a known function is called a Taylor series, see later, e.g. $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$

Power Series

- Recall the sum of a geometric series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$
- There are then many more power series from this formula, e.g. $a = 1$ and $r = x$, then $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

Example

(a) Find a formula for the sum of the following series:
$x^2 + 2x^5 + 4x^8 + 8x^{11} + 16x^{14} + \cdots$
(b) Find a power series representation for $\frac{x}{1-3x}$

Taylor Series

- Many known functions can be expressed in series and they have many applications in engineering

$$
\begin{aligned}
e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \\
\cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \\
\sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \\
\tan^{-1} x &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots \\
\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots
\end{aligned}
$$

Taylor Series

- You can differentiate or integrate power series term-by-term, similar to a polynomial

$$\frac{d}{dx}\left[c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots\right] = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$$
$$\int \left(c_0 + c_1 x + c_2 x^2 + \cdots\right) dx = \left(c_0 x + \frac{c_1}{2} x^2 + \frac{c_2}{3} x^3 + \cdots\right) + C$$

Example

Find a Taylor series for $ln(1 + x)$

Taylor Series

- For gerneral taylor series, let $f(x)$ be a function and suppose that $f(x)$ is analytic (i.e. can be represented by a power series) then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

Example

(a) Assuming that e^x is analytic, find the Taylor series for e^x

(b) Assuming that e^x is analytic, find the Taylor series for $sin\, x$